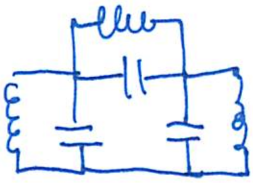
CIRCUIT QUANTISATION

MOTIVATION



For small U and I, expected to behave as a quantum system.
Can this be the case in a circuit of macroscopic size? Yes! For low T.

Fundamental motivation: QM on macroscopic scale. Genuine Q effect, not just QM determining material constitutive equations. (Most magnets & SC devices admit desc in terms of effective classical theories.)

Practical motivation: QT, QC, simulators.

CQED: quantization of E and B fields in terms of dynamic variables adapted to a circuit, same philosophy as in "canonical quantization". Result: microwave photons!
C = circuit

Similarity to cavity QED: JJ ⟺ Rydberg atoms, circuit ⟺ cavity.

Macroscopic quantum tunneling, Leggett (1980-1981), experiment by Martinis, Devoret, Clarke (1981, paper in 1985), escape from washboard potential.

⇒ STANDARD FORMULATION: Node flux quantization

ALTERNATIVES: loop charges, symplectic geometry approach

Note: I'll be using SI units.

Electrical <u>network</u>: connected electrical components/elements

Electrical <u>circuit</u>: network with a closed loop

Linear vs nonlinear: does superposition of signals apply?

Idealized capacitors and inductors are linear. JJs and QPSs are not.

Lumped-element vs. distributed-element network*: lumped in one place, just interconnected points (graphs!); fails for high frequencies or very large circuits (10 GHz ⟺ 3 cm)

Conservative vs. dissipative: presence of resistance vs. lossless

Passive vs. active: (in)capable of power gain (transistors, amplifiers)

Resistor: non-conservative passive

Capacitor, inductor: conservative passive

Voltage/current source, amplifier: active

* We usually work in dual picture: elements are branches/<u>edges</u>, connection points are nodes/<u>vertices</u>.

Electric current : flow of charge through a surface

$$I = \frac{Q}{t}$$

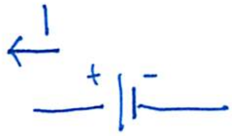
ampere, A

current density

$$j = \lim_{A \to 0} \frac{I_A}{A}$$

create magnetic fields : $\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$

Direction:



Lichtenberg, electrical breakdown, branching figures (1777)

first professor of experimental physics (Göttingen)

Some reference direction needs to be chosen.

charge is conserved. Gauge invariance is not broken in the superconducting state.

Voltage, electrical potential difference, electric tension

$$U = \phi_B - \phi_A \qquad \phi = -\int_C \vec{E}\cdot d\vec{r}, \quad C \text{ arbitrary (electrostatics)}$$

positively charge objects flows towards lower voltage

WARNINGS:

1) more subtle in presence of time-varying fields, gauge freedom

$$\int \vec{E}\cdot d\vec{r} \text{ is path dependent}, \quad \nabla\times\vec{E} \neq 0 \qquad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Need to introduce vector potential $\vec{A}(t)$ to ensure gauge invariance.

2) Voltmeter measures electrochemical potential (Fermi level in SS physics).

3) Warning: conflicting terminology.

η

electrochemical potential $= \mu_{chemical} - e\phi$

$$\left.\begin{aligned} \vec{E} &= -\nabla\varphi \\ \vec{\xi} &= -\nabla\frac{\eta}{q} \\ \eta &= \mu + q\varphi \end{aligned}\right\}$$

see my lecture notes from solid state theory (Boltzmann eq.)

$$\vec{F} = q\vec{E} - \nabla\mu - (\varepsilon_k - \mu)\frac{\nabla T}{T}$$

$$\vec{F} = q\vec{\xi} - (\varepsilon_k - \mu)\frac{\nabla T}{T}$$

Kirchhoff's circuit laws: follow from Maxwell's equations in the low-frequency limit (stationary state)

1st law (junction rule): $\sum I_i = 0$ Algebraic sum of currents meeting at a point is zero.

Conservation of charge.

2nd law (loop rule): $\sum V_i = 0$ Directed sum of voltages around closed loop is zero.

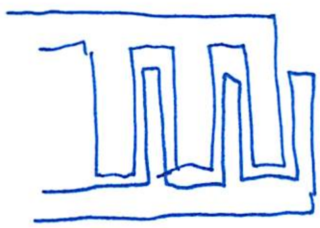
Follows from $\nabla \times \vec{E} = 0$

Capacitors (condenser → kondenzator) —||—

Stores energy in electrical field by storing charge on plates.

capacitance: $C = \frac{Q}{U}$ [F]

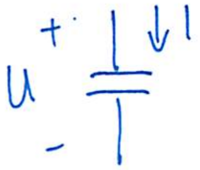
Parallel plates: $C = \varepsilon\varepsilon_0 \frac{A}{d}$ $\varepsilon_0 = 8{,}854 \cdot 10^{-12}$ F/m



interdigitated capacitor

No simple formula, depends on geometric factors, dielectric constant of substrate, thickness of substrate. On-line tools. FEM packages.
COMSOL, ANSYS

$$E = \int_0^Q U(q)\,dq = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CU^2$$



$$I = \frac{dQ}{dt} = C\frac{du}{dt}$$

$$E(t) = \int_{-\infty}^{t} U(t')\,I(t')\,dt'$$

Inductors (coil, choke)

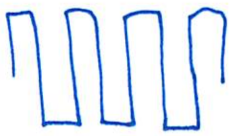
Stores energy in magnetic field when electric current flows.

Inductance: $L = \frac{U}{dI/dt}$ [H] $\Rightarrow$ $U = + L \frac{dI}{dt}$

− or +, depending on conventions

Lenz's law: induced voltage opposes the change in current.

Inductance opposes changes in current.



meandering inductor → magnetic self-inductance

Empirical formulas, on-line calculators (e.g. PCB companies)

Magnetic flux $\Phi_B = \int \vec{B} \cdot d\vec{S}$

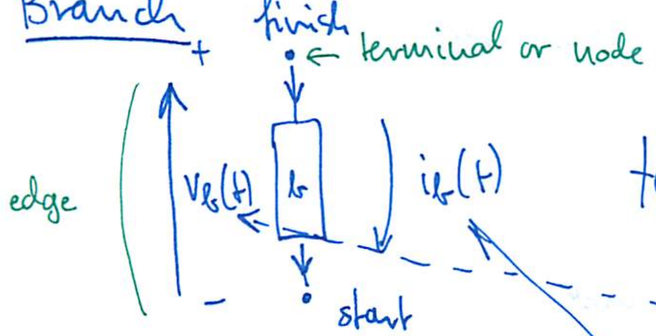
Faraday's law: $U = - \frac{d\Phi_B}{dt}$

Maxwell-Faraday equation: $\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$

$$U = \oint_{\partial \Sigma} \vec{E} \cdot d\vec{r} = - \int_{\Sigma} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = - \frac{d\Phi_B}{dt}$$

$$E = \int U I \, dt = \int L \frac{dI}{dt} I \, dt = \frac{1}{2} L I^2$$

Branch



two-terminal element

$$V_b = \int_{\text{start}}^{\text{end}} \vec{E} \cdot d\vec{r}$$

$$i_b = \frac{1}{\mu_0} \oint_{\text{around } b} \vec{B} \cdot d\vec{S}$$

Natural paths:
- outside of wire for inductors
- outside the dielectric for capacitors

Includes effects of time-varying B and displacement currents!!

$V_b = V_{finish} - V_{start}$

branch voltage ↑ node voltages

generalized flux

(branch-flux Φ_b : $V_b = \frac{d\Phi_b}{dt}$ $\qquad \Phi_b(t) = \int_{-\infty}^{t} V_b(t')\,dt'$

$[W_b] = [V \cdot s]$

$\Phi_b = \Phi_{finish} - \underbrace{\Phi_{start}}_{\text{node fluxes}}$

Energy is stored in branches.
Can be expressed in terms of Φ_b and $\dot{\Phi}_b$.

Nodes and edges define a (directed) graph.

Inductor: $V_b = L \frac{di_b}{dt} = \frac{d\Phi_b}{dt}$

↳ $\Phi_b = L\, i_b$

$E = \frac{\Phi_b^2}{2L}$

Capacitor: $E = \frac{1}{2} C \dot{\Phi}_b^2$

Generally: capacitive element if $V_b = f(Q_b)$

inductive element if $i_b = g(\Phi_b)$

Warning: branch fluxes and voltages are not independent variables (Kirchhoff's laws!).

KIRCHHOFFS LAWS in terms of Φ and Q:

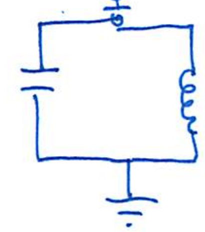
$$\sum_{\substack{\text{all } b \text{ around} \\ \text{loop } l}} \Phi_b = \tilde{\Phi}_l \leftarrow \text{flux through the loop}$$

$$\sum_{\substack{\text{all } b \text{ arriving} \\ \text{at node } n}} Q_b = \tilde{Q}_n \leftarrow \text{charge of the node}$$

Two methods:
- node voltages $\Leftarrow$ more suitable for circuits involving JJs
- loop currents

LC oscillator

Φ



$\Phi_b = \Phi - \Phi_{ground}$ (reference node)

Usually $\Phi_{ground} = 0$.

$E_k = \frac{C}{2}\dot{\Phi}^2 \qquad E_p = \frac{1}{2L}\Phi^2$

Convexity $\frac{\partial^2 L}{\partial \dot{\Phi}^2} > 0$ is not guaranteed. Related to:
- well-posedness of Legendre transform to H
- positivity of kinetic energy
- invertibility of $Q = \frac{\partial L}{\partial \dot{\Phi}}$

Lagrangian:

$$\mathcal{L} = E_k - E_p = \frac{C}{2}\dot{\Phi}^2 - \frac{1}{2L}\Phi^2$$

$\mathcal{L}$ is a convex function of $\dot{\Phi}$.

Conjugate variable: $Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$ $\qquad H = Q\dot{\Phi} - \mathcal{L}$

Hamiltonian: $H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$

Euler-Lagrange: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0$

Q: momentum
Φ: position
C: mass
L^{-1}: spring constant

$C\ddot{\Phi} + \frac{\Phi}{L} = 0 \qquad \ddot{\Phi} + \omega_r^2 \Phi = 0$

$\omega_r = \frac{1}{\sqrt{LC}}$.

Canonical quantization: $Q \to \hat{Q}$, $\Phi \to \hat{\Phi}$ $\qquad [\hat{\Phi}, \hat{Q}] = i\hbar$

Dimensionless: $\phi = 2\pi \frac{\Phi}{\Phi_0} \qquad q = \frac{Q}{2e_0} \qquad \Leftrightarrow [X, P] = i\hbar$

$\Phi_0 = \frac{h}{2e_0} \approx 2 \cdot 10^{-15}$ Wb sc flux quantum

$$H = 4E_C q^2 + \frac{E_L}{2}\phi^2$$

$[\phi, q] = i$

$E_C = \frac{e_0^2}{2C}$ charging energy $\qquad E_L = \frac{\Phi_0^2}{4\pi^2 L}$ inductive energy

$$|\psi\rangle = \int \psi(\phi)\,|\phi\rangle \, d\phi \qquad \ell^2(\mathbb{R})$$

$$\hat{a} = \frac{1}{\sqrt{2L\hbar\omega_r}}\hat{\Phi} + i\frac{1}{\sqrt{2C\hbar\omega_r}}\hat{Q}$$

$$H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) \qquad [a, a^\dagger] = 1$$

voltage ↔ position

$$\Phi = \Phi_{zpf}\,(a^\dagger + a)$$

$$Q = i\,Q_{zpf}\,(a^\dagger - a)$$

current ↔ velocity

Zero-point fluctuations:

$$\Phi_{zpf} = \sqrt{\frac{\hbar}{2C\omega_r}} \qquad Q_{zpf} = \sqrt{\frac{\hbar C\omega_r}{2}}$$

$$\langle 0|(\Delta\hat{\Phi})^2|0\rangle = \Phi_{zpf}^2$$

$$(\Delta\Phi)^2(\Delta Q_{zpf})^2 \geq \frac{\hbar^2}{4}$$

Characteristic impedance: $Z_0 = \sqrt{\frac{L}{C}}$

High Z_0: large V swings

Low Z_0: large I swings

$$\Phi_{zpf} = \sqrt{\frac{\hbar Z_0}{2}} \qquad U = Z_0 I$$

$$Q_{zpf} = \sqrt{\frac{\hbar}{2Z_0}}$$

Finite temperatures:

$$\langle a^\dagger a\rangle = \frac{1}{e^{\beta\hbar\omega_r} - 1} = \frac{1}{2}\coth\frac{\beta\hbar\omega_r}{2} - \frac{1}{2} = n_B(\omega_r)$$

$$\langle a a^\dagger\rangle = n_B(\omega_r) + 1$$

$$\langle\Phi(t)\Phi(0)\rangle = \frac{\hbar Z_0}{2}\left(\langle a^\dagger a\rangle e^{i\omega_r t} + \langle a a^\dagger\rangle e^{-i\omega_r t}\right) = \frac{\hbar Z_0}{2}\left[\coth\left(\frac{\beta\hbar\omega_r}{2}\right)\cos\omega_r t - i\sin\omega_r t\right]$$

$$\langle\Phi^2\rangle = \frac{\hbar Z_0}{2}\coth\frac{\hbar\beta\omega_r}{2}$$

$\to \hbar Z_0/2 \quad T = 0$

$\to k_B T C \quad k_B T \gg \hbar\omega_r$

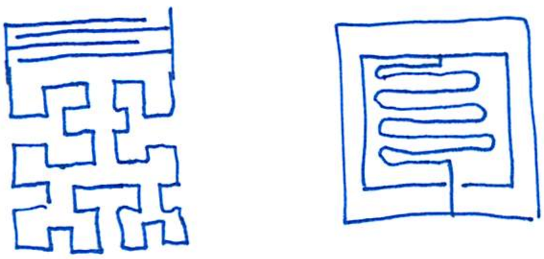
$$\frac{Z_0}{\omega_r} = C$$

imaginary part! OK, because not directly measurable.

Microwave resonators: mm-scale, $2\pi\omega_r \sim 4-10\,\mathrm{GHz}$

Substrate: silicon, sapphire Al, Ni ($T_c = 1{,}2\mathrm{K}, 9\mathrm{K}$)

Photolithographic fabrication



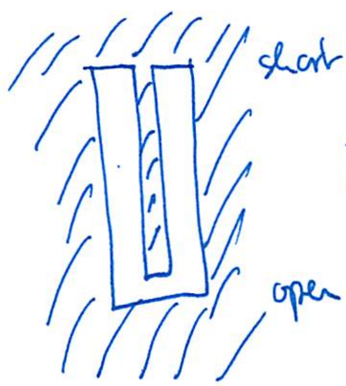
Lumped element

Typical values: $L = 1\,\mathrm{nH}$

$C = 10\,\mathrm{pF}$

$\frac{\omega_0}{2\pi} \simeq 1.6\,\mathrm{GHz}$

$\lambda = 18.8\,\mathrm{cm}$

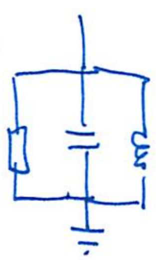


Distributed element

$\lambda/4$

Quantum regime: $\hbar\omega_r \gg k_B T \Rightarrow T \sim 10\,\mathrm{mK}$

Losses:



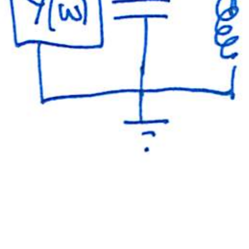
Quality factor: $Q = \omega_r RC = \frac{R}{\omega_r L}$

Rate of energy loss: κ

$\kappa \sim \frac{\omega_r}{Q}$ $\kappa/2\pi = 0.5 - 10\,\mathrm{MHz}$

$\kappa \ll \omega_r$ for quantum regime (width of levels smaller than their separation) $Q \gg 1$

Master equation: $$\frac{d\rho}{dt} = \kappa a \rho a^\dagger - \frac{1}{2}\kappa a^\dagger a \rho - \frac{1}{2}\kappa \rho a^\dagger a$$



admittance

$Y(\omega)$ models EM environment, coupling to measurement leads, etc.

JOSEPHSON EFFECT

Macroscopic quantum phenomenon. Supercurrent that flows continuously without any applied voltage.

JJ: two SC in proximity, barrier or restriction

Josephson equations: ← phase-invariant phase difference, phase difference

$$I(t) = I_c \sin\varphi(t)$$

I_c: critical current, nA-μA range

↳ Ambegaokar – Baratoff $I_c = \frac{\pi\Delta}{2e_0 R_n}$

$$\frac{\partial\varphi}{\partial t} = \frac{2e\,U(t)}{\hbar} = \frac{2\pi}{\Phi_0} U(t)$$

$$\Phi \equiv \Phi_0 \frac{\varphi}{2\pi} \quad \rightarrow \quad U = \frac{d\Phi}{dt}$$

$$\Phi_0 = \frac{h}{2e} = 2.07 \cdot 10^{-15}\,\mathrm{Tm}^2$$

Comments:

A) Relation between flux variables and SC order parameter ($\mathbb{R}$ vs $U(1)$).

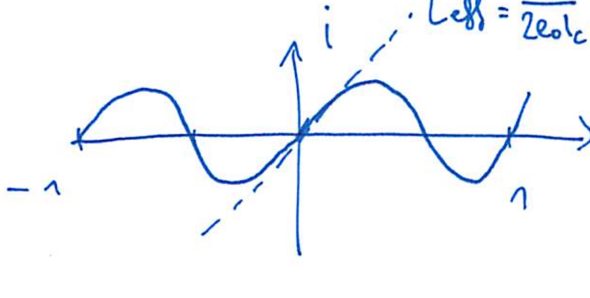
B) No magnetic field, voltage does not come from mag. energy, but from kinetic energy of Cooper pairs.
<u>Kinetic inductance</u>.

$$E = \int IU\,dt = \int I\,d\Phi = \int I_c \sin\varphi\, d\left(\Phi_0 \frac{\varphi}{2\pi}\right) = -\underbrace{\frac{\Phi_0 I_c}{2\pi}}_{E_J}\cos\varphi$$

E_J Josephson energy

 JJ  ≡  real JJ E_C: fF-pF range

$L_{eff} = \frac{\hbar}{2e_0 I_c} \equiv L_{J,0}$



$\frac{\Phi}{\Phi_0} = \frac{\varphi}{2\pi}$

current-flux/phase relation: linear only close to $\varphi \approx 0$

JJ: two pads, Al, Al_2O_3 layer, 1nm $\qquad T_c(Al) = 1.2K$

Feynman's derivation: (optional, če bo čas)

$$\psi = \sqrt{\rho}\, e^{i\theta}$$

$$H = \begin{bmatrix} E_1 & K \\ K & E_2 \end{bmatrix}$$

$$i\hbar\dot{\psi}_1 = E_1\psi_1 + K\psi_2 \quad \rightarrow \quad -\hbar\dot{\theta}_1 + i\frac{\hbar}{2\rho_1}\dot{\rho}_1 = E_1 + K\sqrt{\frac{\rho_2}{\rho_1}}\, e^{i(\theta_2-\theta_1)}$$

$$i\hbar\dot{\psi}_2 = E_2\psi_2 + K\psi_1$$

Im part: $\dot{\rho}_1 = \frac{2K}{\hbar}\sqrt{\rho_1\rho_2}\,\sin(\theta_2-\theta_1)$

current! $\Rightarrow I = I_c \sin\phi$

Re part: $-\dot{\theta}_1 = \frac{E_1}{\hbar} + \frac{K}{\hbar}\sqrt{\frac{\rho_2}{\rho_1}}\cos(\theta_2-\theta_1)$

$$\dot{\theta}_2 = -\frac{E_2}{\hbar} - \frac{K}{\hbar}\sqrt{\frac{\rho_1}{\rho_2}}\cos(\theta_2-\theta_1)$$

$$E_1 - E_2 = 2eV(t)$$

$\rho_1 = \rho_2$

$$\frac{d\phi}{dt} = \frac{2e}{\hbar}V(t) + \frac{K}{\hbar}\underbrace{\left(\sqrt{\frac{\rho_2}{\rho_1}} - \sqrt{\frac{\rho_1}{\rho_2}}\right)}_{0}\cos\phi$$

Independent variables

Graph G, $N+1$ nodes (vertices), M oriented branches (edges)

↑ reference node, $V_0(t) \equiv 0$

(ORIENTED)

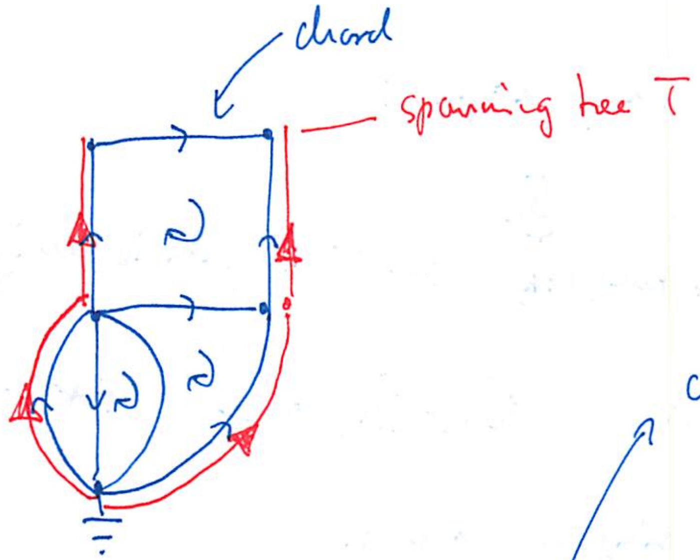
$\underline{\underline{A}}$ $N \times M$ INCIDENCE MATRIX

$$a_{ij} = \begin{cases} +1 & \overset{\bullet}{i} \xrightarrow{j} \quad \text{edge } j \text{ leaves vertex } i \\ -1 & \overset{\bullet}{i} \xleftarrow{j} \\ 0 & \text{otherwise} \end{cases}$$

$$\vec{i}_b = \begin{pmatrix} i_{b1} \\ i_{b2} \\ \vdots \\ i_{bM} \end{pmatrix} \qquad \underline{\underline{A}}\,\vec{i}_b = 0 \qquad \text{Kirchhoff's current law } (1^{st})$$

Each column has one +1 and −1. Generalization to hypergraphs with hyperedges.

Spanning tree: starts from the reference node, reaches all nodes, has no loops. Usually non-unique.



cycle space/cycle basis

chords: $M - N$ $\longleftrightarrow$ # of fundamental $\begin{cases} \text{cycles} \\ \text{loops} \end{cases}$ (1:1 mapping)

~~For~~ planar graphs: clockwise for all loops.

FUNDAMENTAL LOOP MATRIX

$\underline{\underline{B}}$ $(M-N) \times M$

$$b_{ij} = \begin{cases} +1 & \text{branch } j \text{ part of loop } i \text{, same circle} \\ -1 & \text{— " —} \quad \text{; opposite} \\ 0 & \text{otherwise} \end{cases}$$

$$\vec{V}_b = \begin{pmatrix} V_{b1} \\ V_{b2} \\ \vdots \\ V_{b_{M-N}} \end{pmatrix}$$

$\underline{\underline{B}}\ \vec{V}_b = 0$ Kirchhoff's voltage law (2nd)

$\Leftrightarrow \quad \underline{\underline{B}}\ \vec{\Phi}_b = 0$

M-N constraints for M branches: N independent variables!

In presence of external fluxes: $\underline{\underline{B}}\ \vec{\Phi}_b = \vec{\Phi}_{ext}$

↑ fluxes associated w/ each fundamental loop

Idea of ST generalizes to multigraphs: acyclic subgraph in which each except the root has outdegree 1.

$\boxed{\text{Node fluxes}}$ Φ_n, $n = 1, \dots, N$

$$\Phi_n = \sum_{b \in \text{Path}(0 \to n)} \Phi_b \quad \leftarrow \text{This path is unique!}$$

$$n_1 \xrightarrow{b} n_2 \qquad \Phi_{n_2} - \Phi_{n_1} = \Phi_b \qquad \text{For } b \text{ in ST.}$$

For chord closing a loop l:

$$\Phi_{b_l} = \Phi_{n_2} - \Phi_{n_1} + \Phi_{ext,l}$$

convention: + if pointing into the plane

CANONICAL QUANTIZATION

$x \leftrightarrow \Phi \quad p \leftrightarrow q$

(This choice works well for NSs. We could equally choose $x \leftrightarrow q$.)

$$\mathcal{L} = T(\dot{x}_1, \dots, \dot{x}_n) - U(x_1, \dots, x_n)$$

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i} \quad \text{Euler-Lagrange}$$

Gauge transformation: $\mathcal{L} \to \mathcal{L} + \frac{df}{dt}$

Conjugate variables: $p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ $\qquad T = \sum_i m_i \dot{x}_i^2/2 \to p_i = m_i \dot{x}_i$

Hamiltonian is the Legendre transformation of $\mathcal{L}$:

$$H^{(p,x)} = \sup_{\dot{x}_1, \dots, \dot{x}_n} \left(\sum p_i \dot{x}_i - \mathcal{L}(x, \dot{x}) \right)$$

If L convex in $\dot{x}$, supremum occurs at unique point, since $\sum p_i \dot{x}_i - \mathcal{L}$ concave in $\dot{x}_i$.

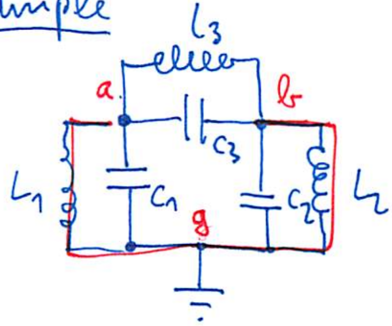
$$\frac{\partial}{\partial \dot{x}_i}\left(\sum p_i \dot{x}_i - \mathcal{L}\right) = 0 \Rightarrow p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right)$$

$$\{x_i, p_j\} = \delta_{ij} \qquad \{x_i, x_j\} = 0 \qquad \{p_i, p_j\} = 0$$

Hamilton's eq:
$$\dot{x}_i = \{x_i, H\} = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial x_i}$$

Example



Spanning tree

Two sets: - branches forming the ST
- branches (chords), each associated w/ fundamental loop

Choosing GND and ST similar to choosing a gauge in EM field theory or choosing the system of position coordinates in classical mechanics.

$$\Phi_{b\in T} = \phi_n - \phi_{n'}$$

$$\Phi_{b\notin T} = \phi_n - \phi_{n'} + \tilde{\Phi}$$

$$\mathcal{L} = \frac{C_1}{2}\dot{\phi}_a^2 + \frac{C_2}{2}\dot{\phi}_b^2 + \frac{C_3}{2}(\dot{\phi}_a - \dot{\phi}_b)^2$$
$$-\left(\frac{1}{2L_1}\phi_a^2 + \frac{1}{2L_2}\phi_b^2 + \frac{1}{2L_3}(\phi_a - \phi_b + \tilde{\Phi})^2\right)$$

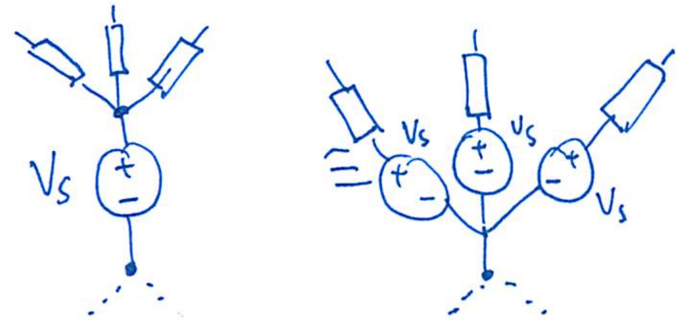
$$q_n = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n} \qquad q_a = C_1\dot{\phi}_a + C_3(\dot{\phi}_a - \dot{\phi}_b)$$

$$q_b = C_2\dot{\phi}_b - C_3(\dot{\phi}_a - \dot{\phi}_b)$$

$$\dot{\phi}_a = \frac{C_2 q_a + C_3(q_a + q_b)}{C_1C_2 + C_1C_3 + C_2C_3} \qquad \dot{\phi}_b = \frac{C_1 q_b + C_3(q_a + q_b)}{C_1C_2 + C_1C_3 + C_2C_3}$$

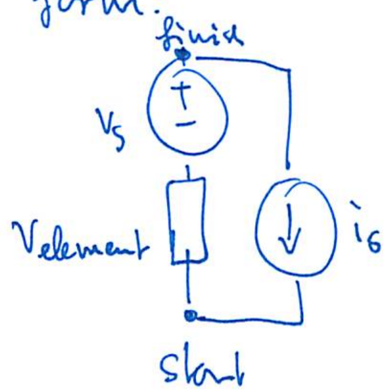
$$H = q_a\dot{\phi}_a + q_b\dot{\phi}_b - \mathcal{L}$$
$$= \frac{1}{2}\frac{1}{C_1C_2 + C_1C_3 + C_2C_3}\left(C_2 q_a^2 + C_1 q_b^2 + C_3(q_a + q_b)^2\right)$$
$$-(\ \dots\ )$$

Current and voltage sources

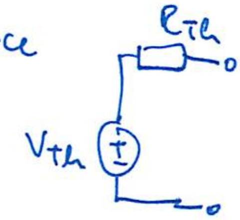


we add countercurrent sources in parallel

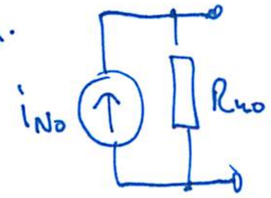
General form:



Thévenin's theorem: linear electrical network can be replaced by a voltage source w/ a series resistance (originally derived by Helmholtz)



Norton's theorem:



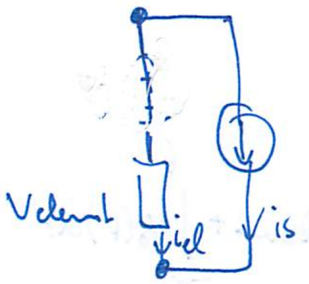
$R_{Th} = R_{No}$

$V_{Th} = I_{No} R_{No}$

$\frac{V_{Th}}{R_{Th}} = I_{No}$

$H = \frac{(Q + C_g V_s)^2}{2C_\Sigma} + \frac{1}{2L}\Phi^2$

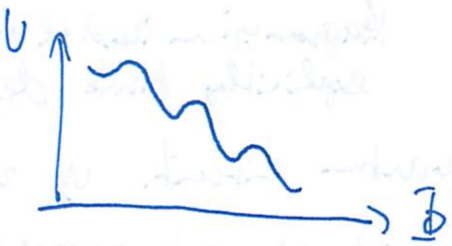
Current source



$i_b = i_s + i_{element}$

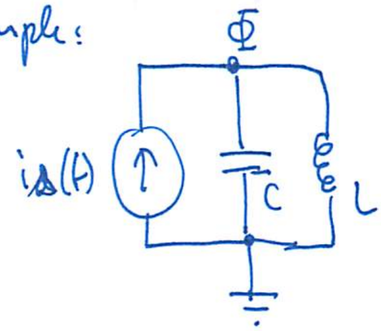
For L: $U = \int_{-\infty}^{t} dt' \, v_b(t') \, i_b(t') = i_s \int_{-\infty}^{t} dt' \, v_b(t') + U_L = i_s \Phi_b(t) + \frac{1}{2L} \Phi_b^2(t)$

For JJ: $U = i_s \Phi_b(t) - E_J \cos\left(\frac{2\pi}{\Phi_0} \Phi_b(t)\right)$



tilted washboard potential

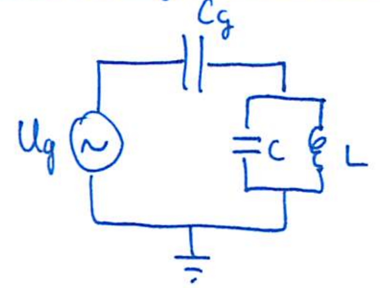
Example:



$$i_s + C\ddot{\Phi} + \frac{\Phi}{L} = 0$$

$$L = \frac{1}{2} C \dot{\Phi}^2 - \left[\frac{1}{2}\frac{\Phi^2}{L} + i_s \Phi\right]$$

Driving the resonator



$$H = \frac{[Q + C_g U_g(t)]^2}{2C_\Sigma} + \frac{\Phi^2}{2L} \qquad C_\Sigma = C_g + C$$

$$H = \frac{Q^2}{2C_\Sigma} + \frac{\Phi^2}{2L} + \frac{C_g}{C_\Sigma} U_g(t)\,\hat{Q}$$

$$U_g(t) = U_0\,\varepsilon(t)\cos(\omega_d t + \phi_0)$$

U_0: amplitude, $\varepsilon(t)$: profile, ω_d: drive frequency, ϕ_0: phase

Gaussian pulse:

$$\varepsilon_g(t) = e^{-\frac{1}{2\sigma^2}(t-t_0)^2}$$

$$H = \hbar\omega_r a^\dagger a + \frac{C_g}{C_\Sigma} V_0\,\varepsilon(t)\cos(\omega_d t + \phi_0)\left[-iQ_{zpf}(a - a^\dagger)\right]$$

$$= \hbar\omega_r a^\dagger a - i\hbar\lambda(t)\left(e^{i(\omega_d t + \phi_0)} + e^{-i(\omega_d t + \phi_0)}\right)(a - a^\dagger)$$

Rotating-frame: $H_I(t) = R(t)H(t)R^{-1}(t) - i\hbar R(t)\frac{d}{dt}R^{-1}(t)$

↑ interaction picture

$$R(t) = e^{i\omega_r a^\dagger a} \qquad \omega_d \approx \omega_r$$

$$H_I(t) = -i\hbar\lambda(t)\Big(a e^{i\phi_0} e^{i(\omega_d - \omega_r)t} - a^\dagger e^{-i\phi_0} e^{-i(\omega_d - \omega_r)t}$$
$$+ a\,e^{i(2\omega_r t + \phi_0)} - a^\dagger e^{-i(2\omega_r t + \phi_0)}\Big)$$

Rotating wave approximation:

$$\alpha(t) = e^{i\phi_0}\int_0^t \lambda(t')\,dt' \qquad |\psi_I(t)\rangle = \underbrace{e^{\alpha(t)a^\dagger - \alpha^*(t)a}}_{\text{displacement operator}}|0\rangle = |\alpha(t)\rangle$$

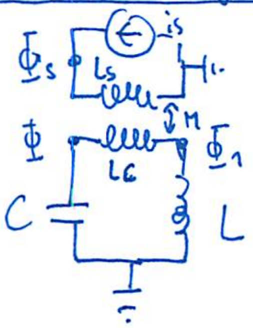
Schrödinger picture: $|\psi(t)\rangle = R^{-1}|\psi_I(t)\rangle = e^{-i\omega_r t}|\alpha(t)\rangle$

Losses: $\alpha(t) = \alpha_0 e^{-\kappa t/2} \Rightarrow$ back to vacuum state in $\approx 1\mu s$

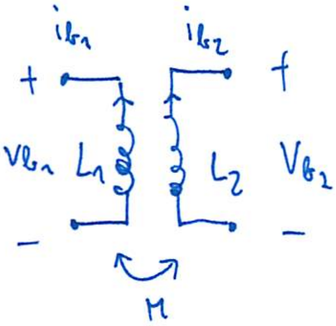
External flux

OPTIONAL, MAY BE OMITTED

18.



$$i_s = \frac{\Phi_s}{L_s}$$



self-inductance

mutual inductance

$$\vec{\Phi}_b = \underline{\underline{M}}\,\vec{i}_b \qquad \underline{\underline{M}} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \qquad \underline{\underline{M}}^{-1} = \frac{1}{L_1L_2 - M^2}\begin{pmatrix} L_2 & -M \\ -M & L_1 \end{pmatrix}$$

$$U_L = \int_{-\infty}^{t} dt'\; \vec{i}_b \cdot \vec{V}_b = \int dt'\; \left(\underline{\underline{M}}^{-1}\vec{\Phi}_b\right)^T \frac{d\vec{\Phi}_b}{dt}$$

$$= \frac{1}{2}\vec{\Phi}_b^{\,T}\,\underline{\underline{M}}^{-1}\,\vec{\Phi}_b = \frac{1}{2(L_1L_2 - M^2)}\left(\Phi_{b1}\;\Phi_{b2}\right)\begin{pmatrix} L_2 & -M \\ -M & L_1 \end{pmatrix}\begin{pmatrix} \Phi_{b1} \\ \Phi_{b2} \end{pmatrix}$$

$M = k\sqrt{L_1L_2}$ $\quad k$: coupling coefficient

$$U = \frac{L_c}{2(L_cL_s - M^2)}\Phi_c^2 + \frac{L_s}{2(L_sL_c - M^2)}\Phi_s^2 - \frac{M}{(L_sL_c - M^2)}\Phi_s\Phi_c$$

$\Phi_c = \Phi_1 - \Phi$ is branch flux across L_c

$M \to 0$ such that $\lim_{M\to 0} M i_s = \Phi_{ext}$ $\qquad \frac{\Phi_s}{L_s} = i_s$

$$U = \frac{1}{2L}\Phi_c^2 + \text{const} - \frac{1}{L_c}\Phi_{ext}\Phi_c = \frac{1}{2L_c}\left(\Phi_c - \Phi_{ext}\right)^2 + \text{const}.$$

$L_c \to 0 \Rightarrow \Phi_c = \Phi_{ext} \rightarrow$ extra branch in the loop accounting for the flux

$$\underline{\underline{B}}\,\vec{\Phi}_b = \vec{\Phi}_{ext}$$

Time-varying fluxes: Important, because used for modulation of qubits (e.g. fast two-qubit gates with transmons)

vector potential $\vec{A}$

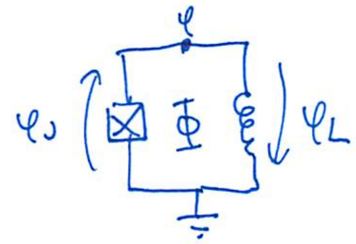
$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E}_B = -\partial_t \vec{A} \qquad \nabla \times \vec{E} = -\partial_t \vec{B}$$

Peierls construction: $V_{jk}(\phi_j - \phi_k) \to V_{jk}(\phi_j - \phi_k + \phi_{ext}^{jk})$

$$\phi_{ext}^{jk} = \frac{2\pi}{\Phi_0} \int_{jk} d\vec{r} \cdot \vec{A}$$

Theory remains gauge-invariant.

Example: fluxonium: Two possible spanning trees



$$H_1 = 4E_C n^2 - E_J \cos\varphi + \frac{1}{2} E_L (\varphi - \phi)^2 \qquad \phi = 2\pi \frac{\Phi}{\Phi_0}$$

$$H_2 = 4E_C n^2 - E_J \cos(\varphi + \phi) + \frac{1}{2} E_L \varphi^2$$

equivalent under static conditions, $U = e^{-\frac{i}{\hbar} Q \Phi} = e^{-in\varphi}$

⚠ Semi-holonomic constraint ($\begin{cases}\text{time} \\ \text{velocity-dependent}\end{cases}$):

$$\varphi_J + \varphi_L = \phi(t) \qquad \dot{\varphi}_J + \dot{\varphi}_L = \dot{\phi}(t)$$

$$\Rightarrow \begin{cases} H_1(t) = 4E_C n^2 - E_J \cos\varphi + \frac{1}{2} E_L [\varphi - \phi(t)]^2 \\ H_2(t) = 4E_C n^2 \underbrace{- 2en\dot{\Phi}} - E_J \cos(\varphi + \phi(t)) + \frac{1}{2} E_L \varphi^2 \end{cases}$$

incorrect w/o this term! Experimentally demonstrated (2023).

$$\mathcal{L} = -\frac{1}{2L}\Phi_L^2 + E_J \cos\left(2\pi \frac{\Phi_J}{\Phi_0}\right) + \frac{1}{2} C \dot{\Phi}_J^2 \qquad \Phi_L = -\Phi_J + \Phi_{ext}(t)$$

$$= -\frac{1}{2L}\Phi_L^2 + E_J \cos(-\varphi_L + \phi_{ext}) + \frac{1}{2} C (\dot{\Phi}_{ext} - \dot{\Phi}_L)^2 \qquad \Phi_J = -\Phi_L + \Phi_{ext}(t)$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C(\dot{\Phi} - \dot{\Phi}_{ext}) \quad \Rightarrow \quad \dot{\Phi} = \frac{1}{C} Q + \dot{\Phi}_{ext} \qquad \begin{matrix} Q = 2e_0 n \\ \Phi_{ext} = \Phi_0 \end{matrix}$$

$$H = Q\dot{\Phi} - \mathcal{L} = \underbrace{C\dot{\Phi}^2 - C\dot{\Phi}\dot{\Phi}_{ext}}_{\frac{Q^2}{C} + Q\dot{\Phi}_{ext}} - \frac{1}{2}\frac{1}{C}Q^2 + \frac{1}{2L}\Phi_L^2 - E_J \cos(-\varphi_L + \phi_{ext})$$

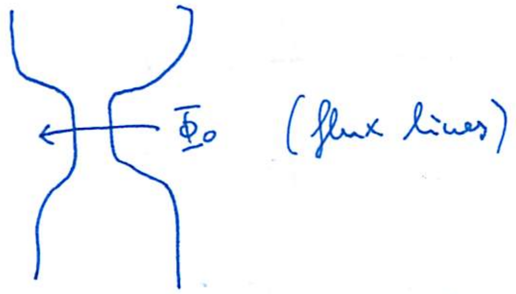
$$E_C = \frac{e_0^2}{2C}$$

$$H = \frac{Q^2}{2C} + \frac{1}{2L}\Phi^2 - E_J \cos(\varphi - \phi_{ext}) + Q\dot{\Phi}_{ext} = 4E_C n^2 + \frac{1}{2} E_L \varphi^2 - E_J \cos(\ldots) + 2e_0 n \dot{\Phi}$$

Ref: Riwar, diVincenzo (2022)
Bryon et al. (2023)
You et al. (2019)

Phase-slip junction

Dual to JJ, (nonlinear) capacitive element, coherent tunneling of flux lines (across a constriction).

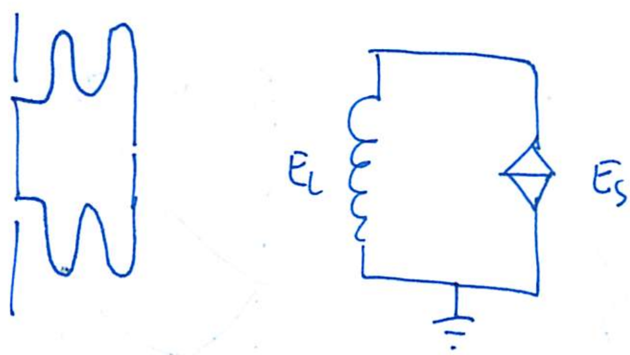


$$H_{QPS} = - E_S \cos(2\pi \hat{q}) + E_L (\hat{\varphi} + \varphi_0)^2$$

Duality: $E_S \leftrightarrow E_J$, $4\pi^2 E_L \leftrightarrow E_C$, $\varphi_0/2\pi \leftrightarrow n_g$, $f \leftrightarrow n$

number of phase slips

Recent experimental breakthrough: Purmessur et al. (2025)



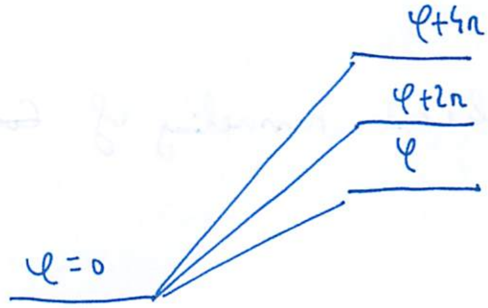
Phase-slip qubit

5nm TiN, large kinetic inductance

17 GHz, 300 mK operation, $T_1 = 50\,\mu s$, $T_2 \sim 10, 20\,\mu s$

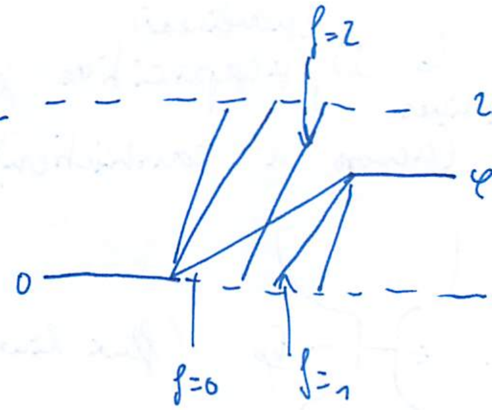
Refs: Mooij, Nazarov
Kaliofoti, Riwar

extended picture



projection $\Longrightarrow$

compact-phase picture



$$H_{QPS} = -\frac{E_S}{2}\sum_f \left(|f\rangle\langle f-1| + |f-1\rangle\langle f| \right) + E_L \sum_f \left(\varphi + 2\pi f \, |f\rangle\langle f|\right)^2$$

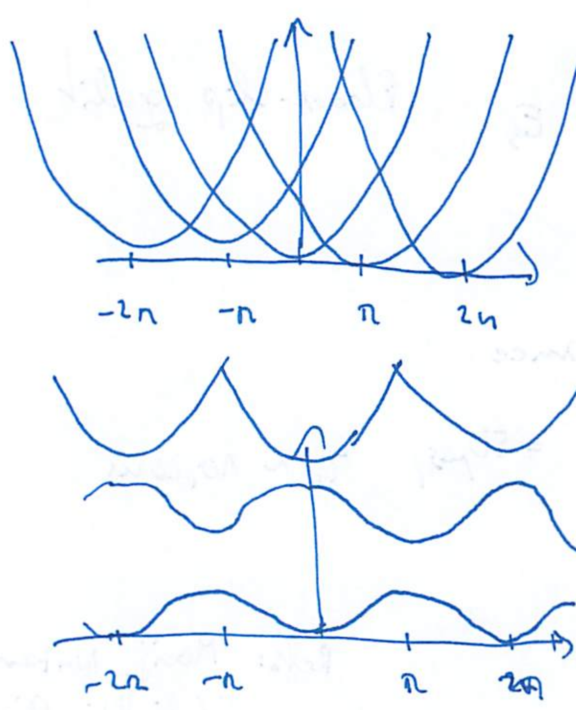
f ← number of kinks

In this formalism, QPS is an inductive element with an internal degree of freedom.

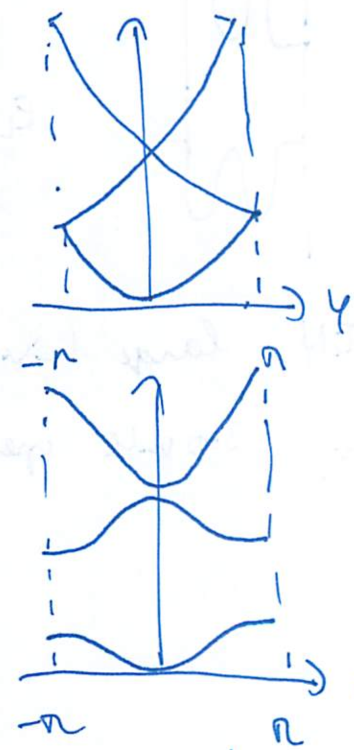
$|f\rangle \to |f\rangle_\varphi$ such that $|f\rangle_{\varphi\pm2\pi} = |f\pm1\rangle_\varphi$

$$H^c_{QPS} = -E_S\cos(\hat{s}_\varphi) + E_L(\varphi+2\pi\hat{f}_\varphi)^2$$

$$e^{i\hat{s}} = \sum_f |f\rangle\langle f-1| \qquad \hat{f} = \sum f\,|f\rangle\langle f| \qquad [e^{i\hat{s}},\hat{f}] = -e^{i\hat{s}}$$

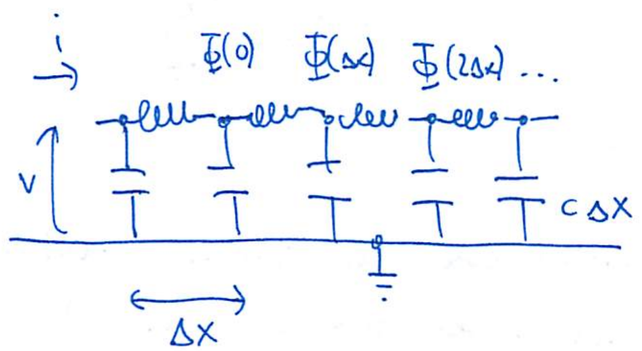


non-compact



compact

Transmission lines and resonators



c, ℓ : capacitance/inductance per unit length

$$L = \sum_{k=-\infty}^{\infty} \frac{c\Delta x}{2} \dot{\Phi}(k\Delta x, t)^2 - \sum_{k=-\infty}^{\infty} \frac{1}{2\ell\Delta x} \left[\Phi((k+1)\Delta x, t) - \Phi(k\Delta x, t) \right]^2$$

$$\Delta x \to 0 \qquad \{\Phi(k\Delta x, t)\} \to \Phi(x,t)$$

$$\sum_k \Delta x\, f(k\Delta x) \to \int_{-\infty}^{\infty} dx\, f(x)$$

Klein-Gordon field $\Phi(x,t)$ w/ Lagrangian

$$L = \int_{-\infty}^{\infty} dx\, \mathcal{L}(\partial_x \Phi, \partial_t \Phi)$$

$$\mathcal{L} = \frac{c}{2}\left(\frac{\partial \Phi}{\partial t}\right)^2 - \frac{1}{2\ell}\left(\frac{\partial \Phi}{\partial x}\right)^2$$

$$i = -\frac{1}{\ell}\frac{\partial \Phi}{\partial x} \qquad v = \frac{\partial \Phi}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \Phi)}\right) = \frac{\partial \mathcal{L}}{\partial \Phi}$$

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{v_p^2}\frac{\partial^2 \Phi}{\partial x^2} = 0 \qquad v_p = \frac{1}{\sqrt{\ell c}}$$

coplanar transmission line on Si substrate, $v_p \approx \frac{c}{2.5}$

$$(\partial_t - v_p \partial_x)(\partial_t + v_p \partial_x)\Phi = 0$$

$$\frac{\partial \Phi}{\partial t} \pm v_p \frac{\partial \Phi}{\partial x} = 0$$

$$\Phi^{\rightarrow} = \Phi^{\rightarrow}(t - x/v_p) \quad \text{right-propagating}$$

$$\Phi^{\leftarrow} = \Phi^{\leftarrow}(t + x/v_p)$$

$$V^{\leftarrow} = - \frac{\partial \Phi^{\leftarrow}}{\partial t}$$

$$V^{\rightarrow} = Z_0 \, i^{\rightarrow} \qquad V^{\leftarrow} = - Z_0 \, i^{\leftarrow}$$

$$Z_0 = \sqrt{\frac{\ell}{c}} \simeq 50\,\Omega$$

$77\,\Omega$ lowest loss (air-filled coaxial)

$30\,\Omega$ max power-transfer (—"—)

$$\Phi(x,t) = \Phi^{\rightarrow}(x,t) + \Phi^{\leftarrow}(x,t)$$

$$= \sqrt{\frac{\hbar}{4\pi c}} \int_{-\infty}^{\infty} dk \, \frac{1}{\sqrt{\omega(k)}} \left(b_k \, e^{-i\omega(k)t + ikx} + b_k^* \, e^{i\omega(k)t - ikx} \right)$$

$$\omega(k) = v_p |k| \qquad [b_k] = [k^{-1/2}] \qquad b_k \to \hat{b}_k \quad b_k^* \to \hat{b}_k^\dagger$$

$$[\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta(k - k')$$

to ensure proper normalization

$$Q(x,t) = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = c \, \frac{\partial \Phi(x,t)}{\partial t}$$

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \left[Q \frac{\partial \Phi}{\partial t} - \mathcal{L} \right] = \int dx \left(\frac{Q^2}{2c} + \frac{1}{2\ell} \left(\frac{\partial \Phi}{\partial x} \right)^2 \right)$$

$$\Phi(x) = \sqrt{\frac{\hbar}{4\pi c}} \int dk \, \frac{1}{\sqrt{\omega}} \left(b_k e^{ikx} + b_k^\dagger e^{-ikx} \right)$$

$$Q(x) = \sqrt{\frac{\hbar c}{4\pi}} \int dk \, \sqrt{\omega} \left(-i b_k e^{ikx} + i b_k^\dagger e^{-ikx} \right)$$

$$[\Phi(x), Q(x')] = i\hbar \, \delta(x - x')$$

Boundaries and resonators

Open-circuit $\quad i(x=0) = -\dfrac{1}{\ell}\dfrac{\partial\Phi}{\partial x} = 0$

$$b_{-k} = -b_k \qquad b_{-k}^* = -b_k^* \quad\Rightarrow\quad \Phi(x,t) = \underbrace{\ }_{\dots}\, 2i\cos(kx)\left(b_k e^{-i\omega t} - b_k^* e^{i\omega t}\right)$$

$$b_k \to \hat{b}_k \qquad b_{-k} \to -\hat{b}_k$$

Grounded: $\quad \Phi(x,t) = \underbrace{\ }_{\dots}\ \sin(kx)\left(b_k e^{-i\omega t} + b_k^* e^{i\omega t}\right)$

$\frac{\lambda}{4}$-resonator: grounded on one side, open on another

Only one mode usually relevant, the one closest to the qubit frequency. $\Rightarrow$ read-out resonator (transmon - $\lambda/4$ - transmission feedline)

Input-output formalism

Walls-Milburn: interakcijska slika, w deli Ω, meje integracije $-\infty$ do $+\infty$ po ω

Semi-infinite TL: $\quad H_{TL} = \displaystyle\int_{-\infty}^{\infty} d\omega\, \hbar\omega\, b_\omega^\dagger b_\omega$

$$H_C = i\hbar\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \underbrace{\sqrt{\kappa(\omega)}}_{\text{coupling strength}}\left(\hat{a}\hat{b}_\omega^\dagger - \hat{a}^\dagger\hat{b}_\omega\right) \qquad \text{capacitive coupling}$$

$$H_{sys} = H_{sys}(\hat{a},\hat{a}^\dagger)$$

κ : decay rate

$\kappa(\omega) \equiv \kappa$: Markovian approximation

$$\frac{d b_\omega}{dt} = \frac{i}{\hbar}\left[H, \hat{b}_\omega\right] = -i\omega\hat{b}_\omega + \sqrt{\frac{\kappa}{2\pi}}\,\hat{a} \qquad t_0 < t < t_1$$

backward $\quad t > t_0 \qquad \hat{b}_\omega(t) = e^{-i\omega(t-t_0)}\hat{b}_\omega(t_0) + \sqrt{\dfrac{\kappa}{2\pi}}\displaystyle\int_{t_0}^{t} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,d\tau$

forward $\quad t_1 > t \qquad \hat{b}_\omega(t) = e^{-i\omega(t-t_1)}\hat{b}_\omega(t_1) - \sqrt{\dfrac{\kappa}{2\pi}}\displaystyle\int_{t}^{t_1} e^{-i\omega(t-\tau)}\hat{a}(\tau)\,d\tau$

input field $\quad \hat{b}_{in}(t) = -\sqrt{\dfrac{1}{2\pi}}\displaystyle\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t_0)}\hat{b}_\omega(t_0)$

output field $\quad \hat{b}_{out}(t) = \sqrt{\dfrac{1}{2\pi}}\displaystyle\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t_1)}\hat{b}_\omega(t_1)$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}\left[H_{sys}, \hat{a}(t)\right] - \sqrt{\frac{\varkappa}{2\pi}}\int_{-\infty}^{\infty} d\omega\, \hat{b}_\omega(t)$$

$$\int_{t_0}^{t} d\tau\, \delta(t-\tau)\hat{a}(\tau) = \frac{1}{2}\hat{a}(t)$$

Markovian approx. needed at this step

$$\frac{d\hat{a}(t)}{dt} = \frac{i}{\hbar}\left[H_{sys}, \hat{a}(t)\right] - \frac{\varkappa}{2}\hat{a}(t) + \sqrt{\varkappa}\, \hat{b}_{in}(t)$$

Quantum stochastic differential equation for resonator mode $\hat{a}(t)$

quantum noise in the input field

$$\frac{d\hat{a}(t)}{dt} = \frac{i}{\hbar}\left[H_{sys}, \hat{a}(t)\right] + \frac{\varkappa}{2}\hat{a}(t) - \sqrt{\varkappa}\, \hat{b}_{out}(t)$$

$$\Rightarrow \quad \hat{b}_{in}(t) + \hat{b}_{out}(t) = \sqrt{\varkappa}\, \hat{a}(t)$$

$$\cancel{\int_{0}^{\infty} e^{ikx}\, dk = \pi\delta(x) + i\, P\left(\frac{1}{x}\right)}$$

$$\underbrace{\int_{-\infty}^{\infty} e^{ikx}\, dk = 2\pi\, \delta(x)}$$

Ref: Walls, Milburn: Quantum optics

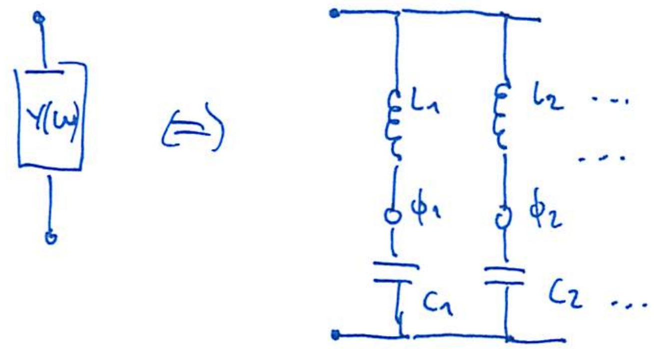
P.V. becomes a one-sided limit, may diverge.

Ref: Walls, Milburn quantum optics

DISSIPATIVE ELEMENTS

Impedance $Z(\omega)$, admittance $Y(\omega) = \frac{1}{Z(\omega)}$.

Dissipation requires a description in terms of a continuum.



LC oscillators in parallel

$$Y_m(\omega) = \left[-iL_m\omega - \frac{1}{iC_m\omega}\right]^{-1} = \frac{-iC_m\omega}{1-(\omega/\omega_m)^2}$$

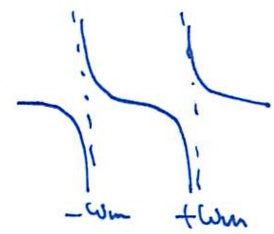
(a) $\omega_m = \frac{1}{\sqrt{L_m C_m}}$

$$i(t) = \int_{-\infty}^{\infty} dt'\, \tilde{Y}(t')\, v(t-t') \qquad Y(\omega) = \int_{-\infty}^{\infty} dt\, \tilde{Y}(t)\, e^{i\omega t - \eta t}$$

$\eta \to 0$

$$Y_m(\omega) = y_m \left\{ \frac{\pi}{2}\delta(\omega-\omega_m) + \frac{\pi}{2}\delta(\omega+\omega_m) + \frac{i}{2} P\frac{\omega_m}{\omega-\omega_m} + \frac{i}{2}\frac{\omega_m}{\omega+\omega_m} \right\}$$

(b) $y_m = \sqrt{\frac{C_m}{L_m}}$



Dense comb of δ functions:

$\omega_{m\neq 0} = m\,\delta\omega$

$y_{m\neq 0} = \mathrm{Re}\left[Y(m\,\delta\omega)\right]$

(a)+(b)

$L_{m\neq 0} = \frac{1}{y_m\omega_m}$ $\qquad L_0 = \frac{1}{\lim_{\omega\to 0} -i\omega Y(\omega)}$

$C_{m\neq 0} = \frac{y_m}{\omega_m}$

two poles

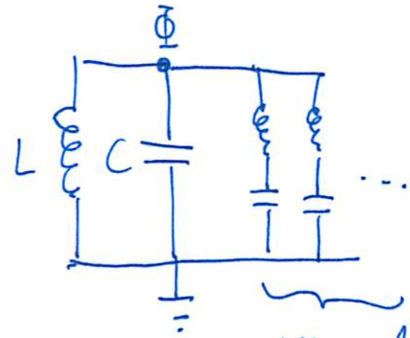
$$Y[\omega + i\eta] = \frac{i}{L_0(\omega+i\eta)} + \lim_{\delta\omega\to 0}\sum_{m=1}^{\infty} Y_m[\omega+i\eta] \qquad \eta > 0$$

This is Caldeira–Leggett model.

Quantum fluctuation-dissipation theorem:

$$\underbrace{\langle \Phi(t)\,\Phi(0)\rangle}_{\text{fluctuations}} = \frac{\hbar}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\left[\coth\left(\frac{\beta\hbar\omega}{2}\right)+1\right]\underbrace{\mathrm{Re}[Z(\omega)]}_{\text{dissipation}}\, e^{-i\omega t}$$

Damped LC oscillator



$$Z_{tot}(\omega)^{-1} = \frac{1}{-iL\omega} - iC\omega + Y(\omega)$$

$$Y(\omega) = \frac{1}{Z(\omega)}$$

QFDT: at $t=0$

$$\langle \Phi^2(t) \rangle = \frac{\hbar Z_0}{2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \coth \frac{\beta\hbar\omega}{2} \, \mathrm{Re}\, Z(\omega)$$

Ohmic case: $Y(\omega) = \dfrac{1}{R + -jL_c\omega} = R^{-1} \dfrac{1}{1 - i\,\omega/\omega_c}$ ← cutoff

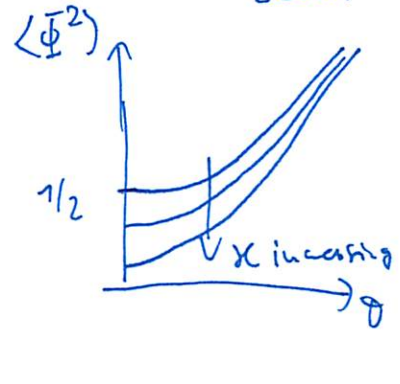
$\omega_c \gg \omega_r$

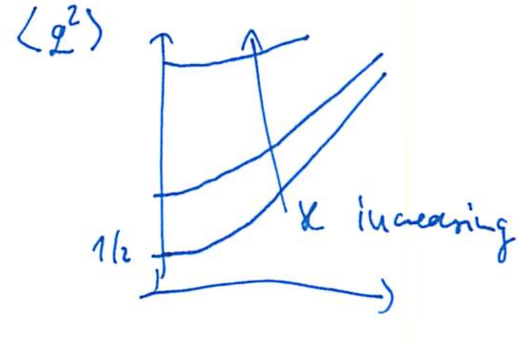
Exact expression in terms of polygamma function

$$\langle \Phi^2 \rangle = \hbar Z_0 \left\{ \frac{k_B T}{\hbar\omega_r} + \frac{1}{2\pi\sqrt{\varkappa^2 - 1}} \left[\Psi(1+\lambda_+) - \Psi(1-\lambda_-) \right] \right\}$$

Damping coefficient:

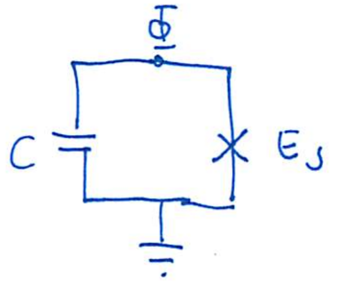
$$\varkappa = \frac{1}{2RC\omega_r} \qquad \lambda_{+/-} = \frac{\varkappa \pm \sqrt{\varkappa^2 - 1}}{2\pi\theta} \qquad \theta = \frac{k_B T}{\hbar\omega_r}$$





Main effect: rescaling!

COOPER PAIR BOX



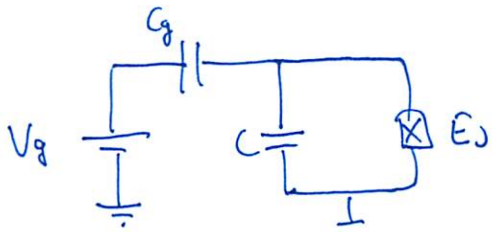
$$\mathcal{L} = \frac{1}{2}C\dot{\Phi}^2 + E_J\cos\left(\frac{2\pi}{\Phi_0}\Phi\right) \Rightarrow C\ddot{\Phi} + I_c\sin\left(\frac{2\pi}{\Phi_0}\Phi\right) = 0$$

Pendulum in a gravitational field

$$\frac{2\pi}{\Phi_0}\Phi = \varphi + 2\pi k \qquad \varphi \in [0; 2\pi) \qquad \text{winding number } k$$

$$\frac{2\pi I_c}{\Phi_0 C} \Leftrightarrow \frac{g}{l}$$

$$H = 4E_C\hat{n}^2 - E_J\cos\phi \qquad \hat{Q} = -i\hbar\frac{\partial}{\partial\phi} = 2e\hat{n}$$



$$H = 4E_C(\hat{n} - n_g)^2 - E_J\cos\phi$$

$n_g = C_gV_g/2e_0$ gate charge

$[\phi, n] = i$

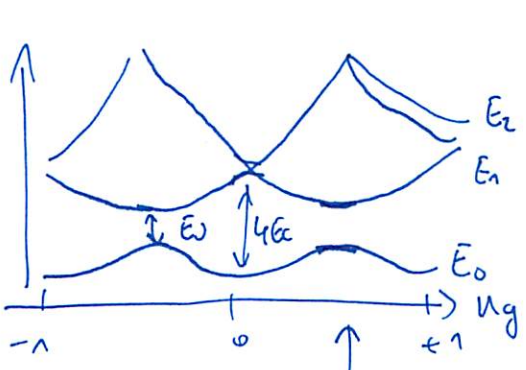
$\hat{n} = -i\frac{\partial}{\partial\Phi}$

COORDINATE REPRESENTATION:

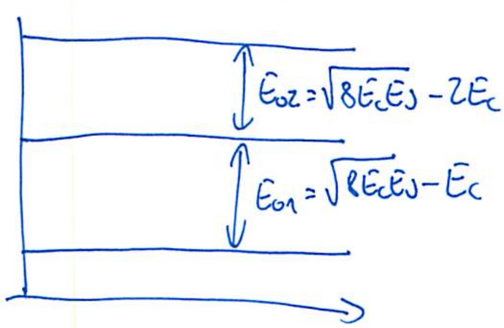
$$4E_C\left(-i\frac{\partial}{\partial\phi} - n_g\right)^2\psi_k(\phi) - E_J\cos\phi\,\psi_k(\phi) = E_k\psi_k(\phi)$$

Analytical solution in terms of Mathieu functions.

$E_J/E_C = 1$

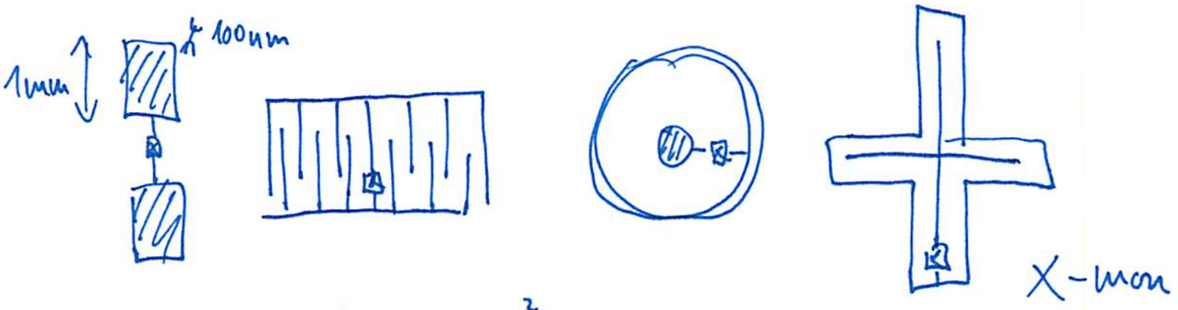


$E_J/E_C = 55$



"sweet spot", derivative ~0

TRANSMON : two SC pads connected by a JJ. $E_J \sim E_C \cdot 50$



JJ area: $100 \times 100\ nm^2$

$C \sim 70-150\ fF$

$$H = 4E_C \hat{n}^2 - E_J \cos\hat{\phi} \simeq 4E_C\hat{n}^2 + \frac{1}{2}E_J\hat{\phi}^2 - \frac{E_J}{4!}\hat{\phi}^4$$

$$\phi = \sqrt{\xi}\,(b + b^\dagger) \qquad \xi = \sqrt{\frac{2E_C}{E_J}}$$

$$\hat{n} = -\frac{i}{2\sqrt{\xi}}(b - b^\dagger)$$

$$H = \sqrt{8E_J E_C}\, b^\dagger b - \frac{E_C}{12}(b + b^\dagger)^4 \Rightarrow E_k = \sqrt{8E_J E_C}\, k - \frac{E_C}{2}(k^2 + k)$$

Anharmonicity $\alpha = \frac{E_{21} - E_{10}}{\hbar} = -E_C/\hbar$

$$H = \omega_q b^\dagger b + \frac{\alpha}{2} b^\dagger b^\dagger b b \qquad \text{Duffing oscillator}$$

$$\frac{\alpha}{12}(b^\dagger + b)^4 = \alpha\left(\frac{1}{2} b^\dagger b^\dagger b b + b^\dagger b\right) + \text{Non. conserving terms}$$

Truncation: to lowest two levels

$b^\dagger - b$	$-i\sigma^y$
$b^\dagger + b$	σ^x
$(b^\dagger - b)^2$	$-\sigma^z$
$(b^\dagger + b)^2$	$-\sigma^z$
$(b^\dagger + b)^3$	$3\sigma^x$
$(b^\dagger + b)^4$	$-6\sigma^z$
	etc.

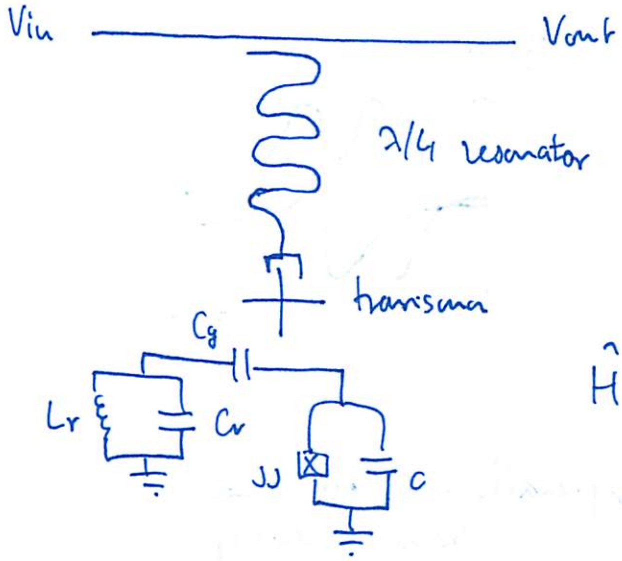
Three levels $\Rightarrow$ qutrits

Circuit QED

↔ cavity QED: Rydberg atoms + ~~optical~~/MW cavity

$\Updownarrow$ qubit (transmon) $\qquad$ $\Updownarrow$ LC oscillator / resonator

Serge Haroche (2012 Nobel prize)



$$\hat{H} = \hbar\omega_r a^\dagger a + 4E_C\left(\hat{n} + \frac{C_g \hat{V}}{2e_0}\right) - E_J\cos\phi$$

$$= \hbar\omega_r a^\dagger a + 4E_C\hat{n}^2 - E_J\cos\phi + \frac{4E_C}{e_0}\hat{n}C_g\hat{V}$$

$$= \hbar\omega_r a^\dagger a + \sqrt{8E_JE_C}\,b^\dagger b - \frac{E_C}{12}(b+b^\dagger)^4$$
$$+ \hbar g(b^\dagger - b)(a - a^\dagger)$$

$$\uparrow$$

$$g = \frac{2E_C}{\hbar e_0}\frac{C_g}{C_r}\frac{Q_{zpf}}{\sqrt{\xi}} = \frac{E_C}{\hbar e_0}\left(\frac{E_J}{2E_C}\right)^{1/4}\frac{C_g}{C_r}\sqrt{2\hbar\omega_r C_r}$$

coupling constant

RWA: $\hat{H} = \hbar\omega_r a^\dagger a + \sqrt{8E_JE_C}\,b^\dagger b - \frac{E_C}{12}(b+b^\dagger)^4 + \hbar g(b^\dagger a + b a^\dagger)$

Restrict to computational basis:

$$\hat{H} = \hbar\omega_r a^\dagger a - \frac{\hbar\omega_{01}}{2}\hat{\sigma}_z + \hbar g(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger)$$

(Jaynes-Cummings H)

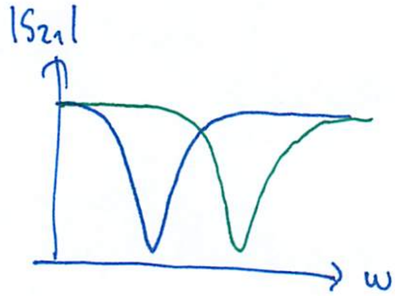
$\Delta = \omega_q - \omega_r$ dispersive regime

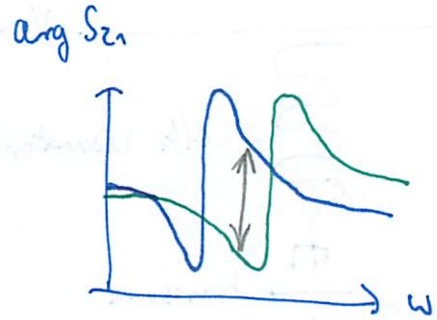
$$H_{disp} = \hbar(\omega_r - \chi\hat{\sigma}_z)a^\dagger a - \frac{\hbar}{2}(\omega_{01} + \chi)\sigma_z \qquad \chi = \frac{g^2}{\Delta}$$

dispersive shift

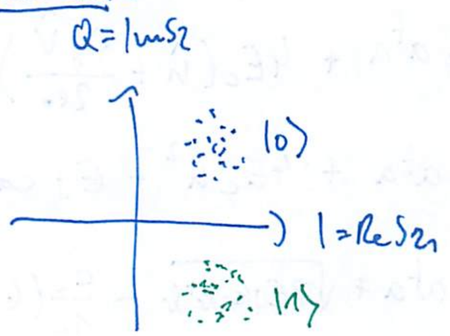
Qubit state can be inferred from a measurement of resonator response.

$V_{out} = S_{21} V_{in}$



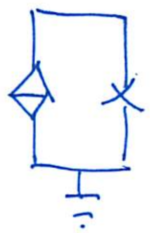


Dispersive <u>read-out</u>:



separation errors due to some overlap

Dualmon qubit (Le et al. PRA 2019)



$$H = -E_Q \cos(2\pi \hat{n}) - E_J \cos\hat{\phi}$$

(difficult to derive, because nonlinearity in both elements)

$$[e^{\pm i\hat{\phi}}, e^{\pm i 2\pi \hat{n}}] = 0 \qquad [\phi, n] = i$$

$$[A, BC] = [A,B]C + B[A,C] \qquad [e^{i\phi}, n] = -e^{i\phi}$$

$$\Rightarrow [\cos(2\pi\hat{n}), \cos\hat{\phi}] = 0$$

Zak basis:

$|k,\varphi\rangle$ $\quad k \in [-1/2; 1/2)$ $\quad$ Double Bloch theorem

$\varphi \in [-\pi; \pi)$

$$\langle k,\varphi | \phi + 2\pi\rangle_{\bar{\phi}} = e^{i2\pi k} \langle k,\varphi | \phi\rangle_{\bar{\phi}} \qquad \langle k,\varphi | k',\varphi'\rangle = \delta(k-k')\delta(\varphi-\varphi')$$

$$\langle k,\varphi | n+1\rangle_{\bar{n}} = e^{i\varphi} \langle k,\varphi | n\rangle_{\bar{n}} \qquad |-1/2,\varphi\rangle = |1/2,\varphi\rangle$$

$$|k,-\pi\rangle = e^{-i2\pi k}|k,\pi\rangle$$

$$E_{k,\varphi} = -E_Q \cos(2\pi k) - E_J \cos\varphi$$

$$|k,\varphi\rangle = \sum_{j=-\infty}^{\infty} e^{i2\pi jk} |\varphi - 2\pi j\rangle_{\bar{\phi}} = \frac{e^{ik\varphi}}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{-ij\varphi} |j-k\rangle_{\bar{n}}$$

Relation to Gottesman-Kitaev-Preskill (GKP) error correcting code

4 critical points: device insensitive to fluctuations in both charge and flux.

$|\bar{0}_{GKP}\rangle = |0,0\rangle$ G.S. $\qquad |\tfrac{1}{2},0\rangle$ saddle point

$|\bar{1}_{GKP}\rangle = |0,\pi\rangle$ saddle point $\qquad |\tfrac{1}{2},\pi\rangle$ maximally excited

LITERATURE

Ciani et al. 2024

Potar : mw

Devoret lecture notes

Manenti, Motta : Quantum information science

Krantz et al. Appl. Phys. Rev. (2019)

Rasmussen et al. (2021)

Girvin : CQED : Superconducting qubits coupled to mw photons
(Oxford Uni. Press) lecture notes

Blais et al. RMP (2021)

Nat. Phys. March 2020 Focus Review Article

CHEAT SHEET

$$\Phi_b = \int_{-\infty}^{t} v_b(t')\,dt' \qquad Q_b = \int_{-\infty}^{t} i_b(t')\,dt'$$

$L: \quad V = L\frac{di}{dt} \qquad \frac{d\Phi}{dt} = L\frac{di}{dt} \quad \Rightarrow \quad \Phi = Li$ $\qquad E^{(L)} = \frac{\Phi^2}{2L} = \frac{1}{2}Li^2$

$C: \quad Q = CV \qquad I = C\frac{dV}{dt} \qquad I = C\ddot{\Phi}$ $\qquad E^{(C)} = \frac{1}{2}C\dot{\Phi}^2$

$R: \quad V = RI \qquad \dot{\Phi} = RI$

$$\Phi_{bET} = \Phi_n - \Phi_{n'}$$

$$\Phi_{b \notin T} = \Phi_n - \Phi_{n'} + \tilde{\Phi}_b$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = \frac{\partial L}{\partial \phi} \qquad q = \frac{\partial L}{\partial \dot\phi} \qquad \dot\phi_n = \frac{\partial H}{\partial q} \qquad \dot q = -\frac{\partial H}{\partial \phi}$$

$$[\phi, q] = i\hbar \qquad [\Phi, Q] = i\hbar \qquad \phi = 2\pi\frac{\Phi}{\Phi_0} \quad q = \frac{Q}{2e_0}$$

$$\Phi_0 = \frac{h}{2e_0}$$

$$\varepsilon_0 \nabla\cdot\vec{E} = \rho \qquad \nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$E_C = \frac{e_0^2}{2C} \qquad E_L = \frac{\Phi_0^2}{4\pi^2 L}$$